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## A sharp upper bound on algebraic connectivity using domination number

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### ABSTRACT

Let  $G$  be a connected graph of order  $n$ . The algebraic connectivity of  $G$  is the second smallest eigenvalue of the Laplacian matrix of  $G$ . A dominating set in  $G$  is a vertex subset  $S$  such that each vertex of  $G$  that is not in  $S$  is adjacent to a vertex in  $S$ . The least cardinality of a dominating set is the domination number. In this paper, we prove a sharp upper bound on the algebraic connectivity of a connected graph in terms of the domination number and characterize the associated extremal graphs.

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## 1. Introduction

Let  $G = (V, E)$  be a simple connected graph of order (number of vertices)  $n$ . The degree of a vertex  $v_i$  is denoted  $d_G(v_i)$  or  $d_i$  when no confusion is possible. The *minimum degree* is denoted by  $\delta$ . The *Laplacian matrix* of  $G$  is defined by  $L = L(G) = D - A$ , where  $D$  is the diagonal matrix which entries are the degrees of the vertices of  $G$ , i.e.,  $D = \text{diag}(d_1, d_2, \dots, d_n)$ , and  $A$  is the *adjacency matrix* of  $G$  defined by  $a_{ij} = 1$  if  $v_i v_j \in E$ , otherwise  $a_{ij} = 0$ . The Laplacian spectrum of  $G$  is the spectrum of  $L(G)$

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and is denoted  $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$  such that  $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ . It is well known that  $\lambda_1 = 0$  and its multiplicity is equal to the number of connected components of  $G$  (see for example [9,10]). The second smallest Laplacian eigenvalue of  $G$ ,  $a = a(G) = \lambda_2$ , is called *algebraic connectivity* of  $G$ . Note that  $a \geq 0$  with equality if and only if  $G$  is not connected [10]. A dominating set in  $G$  is a vertex subset  $S$  such that each vertex of  $G$  that is not in  $S$  is adjacent to a vertex in  $S$ . The least cardinality of a dominating set is the *domination number* and is denoted by  $\beta = \beta(G)$ .

Let  $H$  be a graph on  $k$  vertices  $v_1, v_2, \dots, v_k$ . The *even corona graph* of  $H$ , denoted  $EC(H)$ , is the graph obtained from  $H$  by adding  $k$  vertices  $v'_1, v'_2, \dots, v'_k$  and the edges  $v_i v'_i$  for  $i = 1, \dots, k$ . Note that the number of vertices in  $EC(H)$  is even and equals  $2k$ . The *odd corona graph* of  $H$ , denoted  $OC(H)$ , is the graph obtained from  $H$  by adding  $k - 1$  vertices  $v'_1, v'_2, \dots, v'_{k-1}$  and the edges  $v_i v'_i$  for  $i = 1, \dots, k - 1$ . Note that the number of vertices in  $OC(H)$  is odd and equals  $2k - 1$ . The *pseudo corona graph* of  $H$ , denoted  $PC(H)$ , is the graph obtained from the odd corona graph of  $H$  by adding the edge  $v_k v'_{k-1}$ . Note that the number of vertices in  $PC(H)$  is odd and equals  $2k - 1$ . Let  $PC(H) - e^*$  be the graph obtained from  $PC(H)$  by deleting the edge  $v_{k-1} v_k$ , assuming it exists, i.e., the edge of  $PC(H)$  whose vertices are adjacent to the same vertex  $v'_{k-1}$  that does not belong to the original graph  $H$ .

In [12] the following classes of graphs were defined.

- $\mathcal{G}_1 = \{C_4\} \cup \{G : G \cong EC(H) \text{ where } H \text{ is connected}\}$ , where  $C_4$  denotes the cycle on 4 vertices. Note that the order of each graph in  $\mathcal{G}_1$  is even and if  $G \in \mathcal{G}_1 - \{C_4\}$ , then  $G$  is a spanning graph of  $EC(K_{n/2})$ , where  $n$  is the order of  $G$ .
- $\mathcal{G}_2 = \mathcal{A} \cup \mathcal{B} - \{C_4\}$ , where  $\mathcal{A}$  and  $\mathcal{B}$  are the sets of graphs shown in Figs. 1 and 2 respectively.
- For any graph  $H$ , Let  $\mathcal{S}(H)$  be the set of connected graphs, each of which can be obtained from  $EC(H)$  by adding a vertex  $v$  and edges joining  $v$  to one or more vertices from  $H$ . Then define  $\mathcal{G}_3 = \cup_H \mathcal{S}(H)$ . Note that the order of each graph in  $\mathcal{G}_3$  is odd and if  $G \in \mathcal{G}_3$ , then  $G$  is a spanning graph of  $OC(K_{(n+1)/2})$ , where  $n$  is the order of  $G$ .
- $\mathcal{G}_4 = \{\theta(G) : G \in \mathcal{G}_3\}$ , where  $\theta(G)$  is the graph obtained from  $C_4$  and  $G$  by adding a single edge between a vertex from  $C_4$  and a vertex from  $G$ .
- Consider the path  $P_3 = uvw$  and any graph  $H$ . Let  $\mathcal{P}(H)$  be the set of connected graphs obtained from  $EC(H)$  by joining each of  $u$  and  $w$  to one or more vertices of  $H$ . Then define  $\mathcal{G}_5 = \cup_H \mathcal{P}(H)$ . Note that the order of each graph in  $\mathcal{G}_5$  is odd and if  $G \in \mathcal{G}_5$ , then  $G$  is a spanning graph of  $PC(K_{\frac{n+1}{2}}) - e^*$ , where  $n$  is the order of  $G$ .

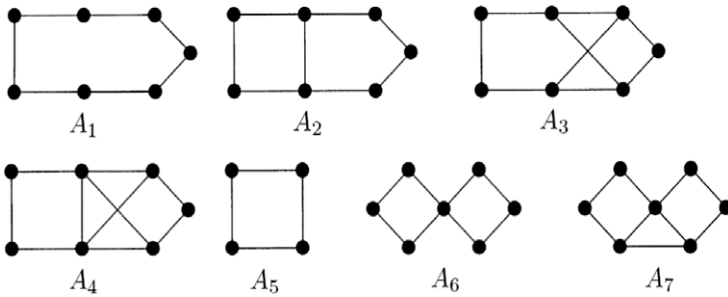


Fig. 1. The graphs of family  $\mathcal{A}$ .

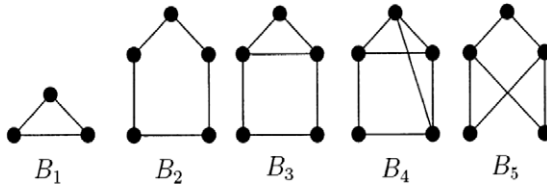


Fig. 2. The graphs of family  $\mathcal{B}$ .

- Let  $H$  be a graph and  $X \in \mathcal{B}$ . Let  $\mathcal{R}(H, X)$  be the set of connected graphs which may be formed from  $EC(H)$  by joining each vertex of  $U \subset V(X)$  to one or more vertices of  $H$  such that no set with fewer than  $\beta(X)$  vertices of  $X$  dominates  $V(X) - U$ . Then define  $\mathcal{G}_6 = \cup_{H, X} \mathcal{R}(H, X)$ .

Finding bounds on the algebraic connectivity has been widely studied (see [1] for references) since it was introduced by Fiedler [10]. In this paper, we are interested in upper bounds on algebraic connectivity in terms of domination number. Such a bound is given in the following theorem.

**Theorem 1** [13]. *If  $G$  is a connected graph on  $n \geq 2$  vertices with algebraic connectivity  $a$  and domination number  $\beta$ , then*

$$a \leq n - \beta + \frac{n - \beta^2}{n - \beta}.$$

For  $\beta \leq \sqrt{n}$  this bound was improved in the next theorem.

**Theorem 2** [14]. *If  $G$  is a connected graph on  $n \geq 2$  vertices with algebraic connectivity  $a$  and domination number  $\beta$ , then*

$$a \leq \begin{cases} n & \text{if } \beta = 1, \\ n - \beta & \text{if } \beta \geq 2. \end{cases}$$

*If  $\beta = 1$  equality holds if and only if  $G \equiv K_n$ . If  $\beta = 2$  equality holds if and only if  $G$  is the complement of a perfect matching. If  $\beta \geq 3$ , the inequality is always strict.*

In this paper, the bounds given in Theorems 1 and 2 are improved in the case  $\beta \geq 3$ .

Since  $a \leq \delta$  for any graph  $G \not\equiv K_n$ , a natural question arises. How tight are the upper bounds, in terms of domination number, on the minimum degree when considered as bounds on the algebraic connectivity? One of these bounds, due to Payan [15], is

$$\delta \leq n - 2\beta + 1.$$

In order to know how tight this bound is if  $\delta$  is replaced by  $a$ , we used AutoGraphiX (a conjecture making system in graph theory [2–6]) to look for extremal graphs for (the graphs that maximize)  $a + 2\beta$  under the constraint  $\beta \geq 3$  (the case  $\beta \leq 2$  is entirely solved by Theorem 2). The “presumably” extremal graphs provided by AutoGraphiX have a regular structure and are well defined by their order. For  $n$  even, the extremal graphs are  $EC(K_{\frac{n}{2}})$  (see Fig. 4 for an example with  $n = 10$ ). If  $n$  is odd, there are three families:  $OC(K_{\frac{n+1}{2}})$ ,  $PC(K_{\frac{n+1}{2}})$  and  $PC(K_{\frac{n+1}{2}}) - e^*$  (see Fig. 4 for examples with  $n = 9$  and 10).

## 2. Preliminary results

In this section, we recall some known results that discuss bounds on the minimum degree of a graph  $G$ , in terms of domination number  $\beta$ . Some results about extremal graphs for given domination number are also given. All these results will be used in the next section.

The following two theorems characterize the graphs of order  $n$  for which the domination number  $\beta = \lfloor n/2 \rfloor$ .

**Theorem 3** [11,16]. *For a graph  $G$  with even order  $n$  and no isolated vertices, the domination number  $\beta = n/2$  if and only if the components of  $G$  are  $C_4$  or the corona graph  $EC(H)$  for any connected graph  $H$ .*

This theorem can be generalized as follows.

**Theorem 4** [12]. *A connected graph  $G$  satisfies  $\beta = \lfloor n/2 \rfloor$  if and only if  $G \in \mathcal{G} = \cup_{i=1}^6 \mathcal{G}_i$ .*

The following theorems provide an upper bound on the domination number  $\beta$  in terms of the number of vertices  $n$  and the minimum degree  $\delta$ .

**Theorem 5** [15,18]. If  $G$  is a connected graph on  $n$  vertices with minimum degree  $\delta$  and domination number  $\beta$ , then

$$\beta \leq \frac{n + 1 - \delta}{2},$$

with exception of the case that  $G$  is the complement of a one-regular graph.

The bound in the above theorem is improved, with few exceptions, in the case of  $\delta \geq 2$  as follows.

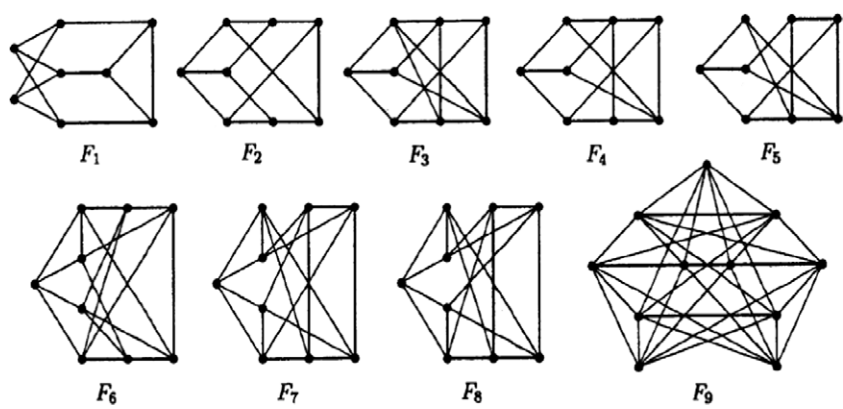
**Theorem 6** [18]. If  $G$  is a connected graph on  $n$  vertices with minimum degree  $\delta \geq 2$ , maximum degree  $\Delta$  and domination number  $\beta$ , then

$$\beta \leq \frac{n - \delta}{2},$$

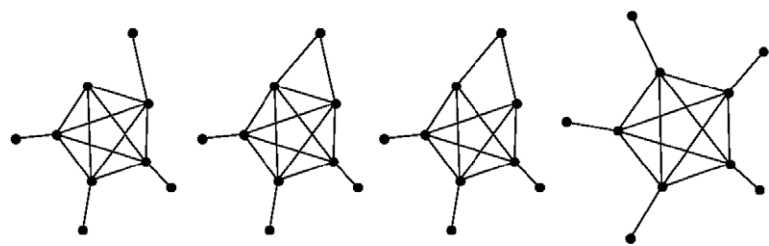
with exception of the cases that  $G$  is either a member of the families  $\mathcal{A}$  (Fig. 1 and Table 1),  $\mathcal{B}$  (Fig. 2) or  $\mathcal{F}$  (Fig. 3), or  $G$  is the complete graph or a graph  $G$  with  $n - 3 \leq \delta \leq \Delta = n - 2$ .

**Table 1**  
Values of  $a$ ,  $\beta$  and  $n$  for the graphs in  $\mathcal{A}$ .

$G$	$A_1$	$A_2$	$A_3$	$A_4$	$A_5$	$A_6$	$A_7$
$a(G)$	0.753020	0.753020	1	1	2	0.585786	0.829914
$\beta(G)$	3	3	3	3	2	3	3
$n$	7	7	7	7	4	7	7



**Fig. 3.** The graphs of family  $\mathcal{F}$ .



**Fig. 4.** Extremal graphs for  $a + 2\beta$  with  $n = 9$  (3, left) and  $n = 10$  (right).

### 3. New results

In this section, new results are proved. The main theorems provide upper bound on the algebraic connectivity  $a$  in terms of the number of vertices  $n$  and the domination number  $\beta$ . Families of extremal graphs are given according to the parity of the number of vertices  $n$ .

First, we prove some results related to the spectra of some graphs defined in Section 1.

**Theorem 7.** Let  $G = (V, E)$  be a graph on  $k$  vertices with Laplacian spectrum  $\lambda_i, i = 1, \dots, k$ .

(i) The Laplacian spectrum of  $EC(G)$  is

$$\mu_i = \frac{\lambda_i \pm \sqrt{\lambda_i^2 + 4}}{2} + 1, \quad i = 1, \dots, k. \quad (1)$$

(ii) If  $\lambda$  is a Laplacian eigenvalue of  $G$  such that the associated eigenvectors  $X = (x_1, x_2, \dots, x_k)^T$  satisfy  $x_k = 0$ , then

$$\mu = \frac{\lambda \pm \sqrt{\lambda^2 + 4}}{2} + 1 \quad (2)$$

are Laplacian eigenvalues of  $OC(G)$ .

(iii) If  $\lambda$  is a Laplacian eigenvalue of  $G$  such that the associated eigenvectors  $X = (x_1, x_2, \dots, x_k)^T$  satisfy  $x_{k-1} = x_k = 0$ , then

$$\mu = \frac{\lambda \pm \sqrt{\lambda^2 + 4}}{2} + 1 \quad (3)$$

are Laplacian eigenvalues of  $PC(G)$ .

**Proof.** (i) For all  $i = 1, \dots, k$ , let  $d_i = d(v_i)$  denote the degree of the vertex  $v_i$  in  $G$ . So  $d_{EC(G)}(v_i) = d_i + 1$  and  $d_{EC(G)}(v'_i) = 1$  for all  $i = 1, \dots, k$ . Let  $\mu \neq 1$  be an eigenvalue of  $EC(G)$  and  $L_{EC(G)}$  the Laplacian matrix of  $EC(G)$ . So, if  $X = (x_1, \dots, x_k, x'_1, \dots, x'_k)^T$  is a  $\mu$ -eigenvector of  $EC(G)$ , we have  $L_{EC(G)}X = \mu X$  or, equivalently

$$\begin{aligned} (d_i + 1)x_i - \sum_{v_j v_i \in E} x_j - x'_i &= \mu x_i, \\ -x_i + x'_i &= \mu x'_i, \end{aligned} \quad i = 1, \dots, k. \quad (4)$$

From the second equation of (4),  $x'_i = x_i / (1 - \mu)$ . Then by substitution in the first equation of (4), we have for every  $i = 1, \dots, k$

$$\begin{aligned} (d_i + 1)x_i - \sum_{v_j v_i \in E} x_j - \frac{x_i}{1 - \mu} &= \mu x_i, \\ d_i x_i - \sum_{v_j v_i \in E} x_j &= \left( \mu - 1 - \frac{1}{\mu - 1} \right) x_i. \end{aligned} \quad (5)$$

Note that Eqs. (5) are the eigenvalue equations for the Laplacian of  $G$ . So

$$\lambda = \left( \mu - 1 - \frac{1}{\mu - 1} \right) \quad (6)$$

is an eigenvalue of  $G$ .

By solving Eq. (6), where  $\mu$  is the unknown and  $\lambda$  is a parameter, the eigenvalues of  $EC(G)$  are of the form

$$\mu_i = \frac{\lambda_i \pm \sqrt{\lambda_i^2 + 4}}{2} + 1, \quad i = 1, \dots, k.$$

Note that, since 0 is a Laplacian eigenvalue of  $G$ , 0 and 2 are Laplacian eigenvalues of  $EC(G)$ .

(ii) Note that  $d_{OC(G)}(v_i) = d(v_i) + 1$  and  $d_{OC(G)}(v'_i) = 1$  for  $i = 1, \dots, k-1$  and  $d_{OC(G)}(v_k) = d(v_k)$ . Let  $\mu \neq 1$  be a Laplacian eigenvalue of  $OC(G)$  and  $X = (x_1, \dots, x_k, x'_1, \dots, x'_{k-1})^T$  a  $\mu$ -eigenvector. Then

$$\begin{aligned} (d_i + 1)x_i - \sum_{v_i v_j \in E} x_j - x'_i &= \mu x_i, \quad i = 1, \dots, k-1, \\ -x_i + x'_i &= \mu x'_i, \\ d_k x_k - \sum_{v_k v_j \in E} x_j &= \mu x_k. \end{aligned} \quad (7)$$

Proceeding as in (i), we get

$$\begin{aligned} d_i x_i - \sum_{v_i v_j \in E} x_j &= \left(\mu - 1 - \frac{1}{\mu - 1}\right) x_i, \quad i = 1, \dots, k-1, \\ d_k x_k - \sum_{v_k v_j \in E} x_j &= \mu x_k. \end{aligned} \quad (8)$$

Note that the equations in (8) are the eigenvalue equations for the Laplacian of  $G$  if and only if  $\mu = 0$  or  $x_k = 0$ . It is obvious that  $\mu = 0$  is a Laplacian eigenvalue of  $OC(G)$ , so consider the case  $x_k = 0$ , in which the equations in (8) become

$$\begin{aligned} d_i x_i - \sum_{v_i v_j \in E} x_j &= \left(\mu - 1 - \frac{1}{\mu - 1}\right) x_i, \quad i = 1, \dots, k-1, \\ \sum_{v_k v_j \in E} x_j &= 0. \end{aligned} \quad (9)$$

The equations in (9) characterize the eigenvalues  $\lambda$  of  $G$  whose eigenvectors  $k$ -th entry is 0. Then by solving  $\mu - 1 - \frac{1}{\mu - 1} = \lambda$ , we get that  $\mu = \frac{\lambda \pm \sqrt{\lambda^2 + 4}}{2} + 1$  are Laplacian eigenvalues of  $OC(G)$ . The multiplicity of each is equal to the dimension of the subspace, of the eigenspace associated to  $\lambda$ , whose eigenvectors  $k$ -th entry is 0.

(iii) Note that  $d_{PC(G)}(v_i) = d(v_i) + 1$  for  $i = 1, \dots, k$ ,  $d_{PC(G)}(v'_i) = 1$  for  $i = 1, \dots, k-2$  and  $d_{PC(G)}(v_{k-1}) = 2$ . Let  $\mu \notin \{1, 2\}$  be a Laplacian eigenvalue of  $PC(G)$  and  $X = (x_1, \dots, x_k, x'_1, \dots, x'_{k-1})^T$  a  $\mu$ -eigenvector. Then

$$\begin{aligned} (d_i + 1)x_i - \sum_{v_i v_j \in E} x_j - x'_i &= \mu x_i, \quad i = 1, \dots, k-2, \\ (d_{k-1} + 1)x_{k-1} - \sum_{v_{k-1} v_j \in E} x_j - x'_{k-1} &= \mu x_{k-1}, \\ (d_k + 1)x_k - \sum_{v_k v_j \in E} x_j - x'_{k-1} &= \mu x_k, \\ -x_i + x'_i &= \mu x'_i, \quad i = 1, \dots, k-2, \\ -x_{k-1} - x_k + 2x'_{k-1} &= \mu x'_{k-1}. \end{aligned}$$

From the last equation, we have

$$x'_{k-1} = -\frac{1}{\mu - 2}x_{k-1} - \frac{1}{\mu - 2}x_k$$

and then, proceeding as in (i) and (ii), we get

$$\begin{aligned} d_i x_i - \sum_{v_i v_j \in E} x_j &= \left(\mu - 1 - \frac{1}{\mu - 1}\right) x_i, \quad i = 1, \dots, k-2, \\ d_{k-1} x_{k-1} - \sum_{v_{k-1} v_j \in E} x_j &= \left(\mu - 1 - \frac{1}{\mu - 2}\right) x_{k-1} - \frac{1}{\mu - 2} x_k, \\ d_k x_k - \sum_{v_k v_j \in E} x_j &= \left(\mu - 1 - \frac{1}{\mu - 2}\right) x_k - \frac{1}{\mu - 2} x_{k-1}. \end{aligned}$$

If  $x_{k-1} = x_k = 0$ , these equations are the eigenvalue equations for the Laplacian of  $G$ . In this case, the equations are

$$\begin{aligned} d_i x_i - \sum_{v_i v_j \in E} x_j &= \left( \mu - 1 - \frac{1}{\mu - 1} \right) x_i, \quad i = 1, \dots, k-2, \\ \sum_{v_{k-1} v_j \in E} x_j &= 0, \\ \sum_{v_k v_j \in E} x_j &= 0. \end{aligned}$$

These equations characterize the eigenvalues  $\lambda$  of  $G$  whose eigenvectors  $(k-1)$ -th and  $k$ -th entries are 0. Then by solving  $\mu - 1 - \frac{1}{\mu - 1} = \lambda$ , we get  $\mu = \frac{\lambda \pm \sqrt{\lambda^2 + 4}}{2} + 1$  are Laplacian eigenvalues of  $PC(G)$ . The multiplicity of each is equal to the dimension of the subspace, of the eigenspace associated to  $\lambda$ , whose eigenvectors  $(k-1)$ -th and  $k$ -th entries are 0.  $\square$

**Corollary 8.** Let  $k \geq 4$  be an integer.

(i) The Laplacian spectrum of  $EC(K_k)$  is

$$\begin{pmatrix} 0 & \frac{k+2-\sqrt{k^2+4}}{2} & 2 & \frac{k+2+\sqrt{k^2+4}}{2} \\ 1 & k-1 & 1 & k-1 \end{pmatrix}.$$

(ii) The Laplacian spectrum of  $OC(K_{k+1})$  is

$$\begin{pmatrix} 0 & \frac{k+3-\sqrt{(k+1)^2+4}}{2} & \frac{k+3-\sqrt{(k-1)^2+4}}{2} & \frac{k+3+\sqrt{(k-1)^2+4}}{2} & \frac{k+3+\sqrt{(k+1)^2+4}}{2} \\ 1 & k-1 & 1 & 1 & k-1 \end{pmatrix}.$$

(iii)  $\frac{k+3-\sqrt{(k+1)^2+4}}{2}$ ,  $\frac{k+3+\sqrt{(k+1)^2+4}}{2}$  and  $k+2$  are eigenvalues of  $PC(K_{k+1})$  with multiplicities  $k-2$ ,  $k-2$  and 1 respectively.

(iv)  $\frac{k+3-\sqrt{(k+1)^2+4}}{2}$ ,  $\frac{k+3+\sqrt{(k+1)^2+4}}{2}$  and  $k$  are eigenvalues of  $PC(K_{k+1}) - e^*$  with multiplicities  $k-2$ ,  $k-2$  and 1 respectively.

**Proof.** (i) The result follows from Theorem 7(i) by replacing  $G$  by  $K_k$ , the spectrum of which is  $\lambda_1 = 0$  and  $\lambda_2 = \dots = \lambda_k = k$ .

(ii) Here we use Theorem 7(ii) by replacing  $G$  by  $K_{k+1}$  whose Laplacian spectrum is  $\lambda_1 = 0$  and  $\lambda_2 = \dots = \lambda_{k+1} = k+1$ . First, we have to show that there are Laplacian eigenvectors associated to  $\lambda = k+1$  such that the  $(k+1)$ th entry of each is 0. Under these conditions the Laplacian eigenvalues equations are

$$\begin{aligned} kx_i - \sum_{v_i v_j \in E} x_j &= (k+1)x_i, \quad i = 1, \dots, k, \\ \sum_{v_{k+1} v_j \in E} x_j &= 0, \end{aligned}$$

which are equivalent to the single equation

$$\sum_{j=1}^k x_j = 0.$$

This equation has exactly  $k-1$  independent solutions, i.e., there are exactly  $k-1$  Laplacian eigenvectors associated to  $\lambda = k+1$  such that the  $(k+1)$ th entry of each is 0. Then by Theorem 7(ii),  $\frac{k+3-\sqrt{(k+1)^2+4}}{2}$  and  $\frac{k+3+\sqrt{(k+1)^2+4}}{2}$  are eigenvalues of  $OC(K_{k+1})$  and the multiplicity of each is  $k-1$ .

Obviously, 0 is a Laplacian eigenvalue of  $OC(K_{k+1})$ . To compute the remaining two eigenvalues, we use

$$\sum_{i=1}^{2k+1} \mu_i = \text{Tr}(L_{OC(K_{k+1})}) = k^2 + 3k,$$

$$\sum_{i=1}^{2k+1} \mu_i^2 = \text{Tr}(L_{OC(K_{k+1})}^2) = k^3 + 3k^2 + 5k.$$

Some easy computations give that the two eigenvalues are  $\frac{k+3-\sqrt{(k-1)^2+4}}{2}$  and  $\frac{k+3+\sqrt{(k-1)^2+4}}{2}$ .

(iii) We proceed exactly as in (ii) above. The only difference is that (since  $x_k = x_{k+1} = 0$ ) the eigenvalue equations are equivalent to

$$\sum_{j=1}^{k-1} x_j = 0$$

for which there are exactly  $k - 2$  independent eigenvectors whose  $k$ th and  $(k + 1)$ th entries are 0.

Then by Theorem 7(iii),  $\frac{k+3-\sqrt{(k+1)^2+4}}{2}$  and  $\frac{k+3+\sqrt{(k+1)^2+4}}{2}$  are eigenvalues of  $PC(K_{k+1})$  and the multiplicity of each is  $k - 2$ .

It is easy to see that the  $k$ th and  $(k + 1)$ th lines (columns) of  $L(PC(K_{k+1})) - (k + 2)I$  are identical. So  $k + 2$  is a Laplacian eigenvalue of  $PC(K_{k+1})$ . Let  $b, c$  and  $d$  be the remaining non-zero eigenvalues (with possible repetitions) of  $PC(K_{k+1})$ . To show that the multiplicity of  $k + 2$  is 1, it suffices to prove that  $k + 2 \notin \{b, c, d\}$ . Indeed, suppose the contrary and let (without loss of generality)  $d = k + 2$ . Then using the relations

$$\text{Tr}(L_{PC(K_{k+1})}^p) = \sum_{i=1}^{2k+1} \lambda_i^p \quad \text{for } p = 1, 2, 3,$$

we get the following equations

$$\begin{aligned} b + c &= 4, \\ b^2 + c^2 &= 12, \\ b^3 + c^3 &= 6k + 34, \end{aligned}$$

which are unsolvable if  $k > 1$ .

(iv) This case is proved exactly like (iii).  $\square$

**Proposition 9.** If  $\lambda \notin \{k, k + 2\}$ , then  $\lambda$  is a Laplacian eigenvalue of  $PC(K_{k+1})$  if and only if  $\lambda$  is a Laplacian eigenvalue of  $PC(K_{k+1}) - e^*$ . In addition, the graphs  $OC(K_{k+1})$ ,  $PC(K_{k+1})$  and  $PC(K_{k+1}) - e^*$  have the same algebraic connectivity  $a = \frac{k+3-\sqrt{(k+1)^2+4}}{2}$ .

**Proof.** The eigenvalue equations of  $PC(K_{k+1})$  and  $PC(K_{k+1}) - e^*$  differ only in the equations corresponding to the vertices  $v_k$  and  $v_{k+1}$ . These two equations for a  $PC(K_{k+1})$  Laplacian eigenvalue  $\lambda \neq k + 2$  are

$$(k + 1)x_k - \sum_{j=1, j \neq k}^{k+1} x_j - x'_k = \lambda x_k, \quad (10)$$

$$(k + 1)x_{k+1} - \sum_{j=1}^k x_j - x'_k = \lambda x_{k+1}. \quad (11)$$



Taking the difference between (10) and (11), we have

$$(k+2)(x_k - x_{k+1}) = \lambda(x_k - x_{k+1}). \quad (12)$$

Since  $\lambda \neq k+2$  and a Laplacian eigenvalue of  $PC(K_{k+1})$ , necessarily  $x_k = x_{k+1}$  and therefore (10) and (11) become

$$kx_k - \sum_{j=1}^{k-1} x_j - x'_k = \lambda x_k,$$

$$kx_{k+1} - \sum_{j=1}^{k-1} x_j - x'_k = \lambda x_{k+1},$$

which are exactly the eigenvalue equations of  $PC(K_{k+1}) - e^*$  corresponding to the vertices  $v_k$  and  $v_{k+1}$ . Then, and since the remaining eigenvalue equations are the same,  $\lambda$  is a Laplacian eigenvalue of  $PC(K_{k+1}) - e^*$ .

Similarly, we can prove that if  $\lambda \neq k$  is a Laplacian eigenvalue of  $PC(K_{k+1}) - e^*$  so it is for  $PC(K_{k+1})$ .

Obviously  $a(PC(K_{k+1})) = a(PC(K_{k+1}) - e^*)$  holds. On the other hand, it follows from Corollary

8(ii) that  $a(OC(K_{k+1})) = \frac{k+3-\sqrt{(k+1)^2+4}}{2}$ . Thus to be done, it suffices to prove that  $a(PC(K_{k+1})) = a(OC(K_{k+1}))$ . Let  $0 = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_{2k+1}$  and  $0 = \lambda'_1 \leq \lambda'_2 \leq \dots \leq \lambda'_{2k+1}$  be the spectra of  $OC(K_{k+1})$  and  $PC(K_{k+1})$  respectively.

Using the Courant–Weyl inequalities (see, e.g., [9, Theorem 2.1]) and the fact that  $PC(K_{k+1}) = OC(K_{k+1}) + v_{k+1}v'_k$ , it follows that

$$0 = \lambda_1 = \lambda'_1 \leq \lambda_2 \leq \lambda'_2 \leq \dots \leq \lambda_{2k+1} \leq \lambda'_{2k+1}.$$

On the other hand, according to Corollary 8(ii), we have  $\lambda_2 = \lambda_3 = \frac{k+3-\sqrt{(k+1)^2+4}}{2}$ . It follows that  $a(PC(K_{k+1})) = \frac{k+3-\sqrt{(k+1)^2+4}}{2}$ .  $\square$

**Lemma 10.** If  $G$  is a connected graph on  $n$  vertices with algebraic connectivity  $a$ , then  $a(EC(G)) \leq \frac{n+2-\sqrt{n^2+4}}{2}$  with equality if and only if  $G \equiv K_n$ .

**Proof.** If  $G \equiv K_n$  the equality follows from Corollary 8(i). Now, let  $G \not\equiv K_n$ , then  $a(G) \leq n-2$ . By Theorem 7(i),  $\frac{a(G)+2-\sqrt{(a(G))^2+4}}{2}$  is an eigenvalue of  $EC(G)$ . On the other hand, the function  $f(t) = t - \sqrt{t^2+4}$  is increasing. Thus  $a < \frac{n+2-\sqrt{n^2+4}}{2}$ .  $\square$

**Theorem 11.** Let  $G$  be a connected graph with even order  $n = 2k \geq 6$ , algebraic connectivity  $a$  and domination number  $\beta \geq 3$ . Then

$$a \leq 2k - 2\beta + \frac{k+2-\sqrt{k^2+4}}{2}$$

with equality if and only if  $G$  is  $EC(K_k)$ .

**Proof**

- If the minimum degree  $\delta \geq 2$  and  $G \notin \{F_1, F_2, \dots, F_5\}$ , then by Theorem 6 and the fact that  $a \leq \delta$  ( $G \not\equiv K_n$ ), we have

$$a + 2\beta \leq \delta + 2\beta \leq n = 2k < 2k + \frac{k+2-\sqrt{k^2+4}}{2}.$$

Thus the bound is not reached in this case.

**Table 2**Values of  $a$ ,  $\beta$  and  $n$  for the graphs in  $\mathcal{F}$ .

$G$	$F_1$	$F_2$	$F_3$	$F_4$	$F_5$	$F_6$	$F_7$	$F_8$	$F_9$
$a(G)$	1.438447	2	2.267949	2	2	3	3	3	4.763932
$\beta(G)$	3	3	3	3	3	3	3	3	3
$n$	8	8	8	8	8	9	9	9	11

- If  $G \in \{F_1, F_2, \dots, F_5\}$ , the bound is true from Table 2.
- If the minimum degree  $\delta = 1$ , then by Theorem 5,

$$a + 2\beta \leq \delta + 2\beta \leq n + 1.$$

If  $\delta + 2\beta \leq n$ , then  $a + 2\beta < \frac{k+2-\sqrt{k^2+4}}{2}$ . If  $\delta + 2\beta = n + 1$ , then there exists a graph  $H$  such that  $G \equiv EC(H)$ . Thus by Lemma 10  $a \leq a(EP(K_k))$  with equality if and only if  $H \equiv K_k$ . Therefore the result follows.  $\square$

**Lemma 12.** Let  $G \in \cup_{i=3}^6 \mathcal{G}_i$  with order  $n = 2k + 1$ .

- (i) If  $G \in \mathcal{G}_3$ , then  $a(G) \leq a(OC(K_{k+1})) = \frac{k+3-\sqrt{(k+1)^2+4}}{2}$ .
- (ii) If  $G \in \mathcal{G}_4$ , then  $a(G) \leq a(EC(K_{k-1})) = \frac{k+1-\sqrt{(k-1)^2+4}}{2}$ .
- (iii) If  $G \in \mathcal{G}_5$ , then  $a(G) \leq a(PC(K_{k+1} - e^*)) = \frac{k+3-\sqrt{(k+1)^2+4}}{2}$ .
- (iv) If  $G \in \mathcal{G}_6$ , then  $a(G) \leq \frac{k+3-\sqrt{(k+1)^2+4}}{2}$ .

### Proof

- (i) It is easy to see that if  $G \in \mathcal{G}_3$  with  $2k + 1$  vertices, then  $G$  is a spanning subgraph of  $OC(K_{k+1})$ . Thus the inequality follows.
- (ii) Let  $H$  be the graph in  $\mathcal{G}_4$  corresponding to  $OC(K_{k-1})$  in  $\mathcal{G}_3$ . Then  $H$  is the union of the graph  $H_1$  composed of  $C_4$  and  $2k - 3$  isolated vertices, and the graph  $H_2$  composed of  $EC(K_{k-1})$  and 3 isolated vertices. The Laplacian spectrum of  $H_1$  is  $\lambda_1(H_1) = \dots = \lambda_{2k-2}(H_1) = 0$ ,  $\lambda_{2k-1}(H_1) = \lambda_{2k}(H_1) = 2$  and  $\lambda_{2k+1}(H_1) = 4$ ; and the Laplacian spectrum of  $H_2$  (using Corollary 8) is  $\lambda_1(H_2) = \dots = \lambda_4(H_2) = 0$  and  $\lambda_5(H_2) = \dots = \lambda_{k+2}(H_2) = \frac{k+1-\sqrt{(k-1)^2+4}}{2}$ ,  $\lambda_{k+3} = 2$  and  $\lambda_{k+4}(H_2) = \dots = \lambda_{2k+1}(H_2) = \frac{k+1+\sqrt{(k-1)^2+4}}{2}$ . Now, using the Courant–Weyl inequalities (see, e.g., [9, Theorem 2.1]) we have
 
$$\lambda_2(H) \leq \lambda_{2k-2}(H_1) + \lambda_5(H_2) = \frac{k+1-\sqrt{(k-1)^2+4}}{2}.$$
- (iii) The inequality follows from the fact that any graph in  $\mathcal{G}_5$  with  $2k + 1$  vertices, is a spanning subgraph of  $PC(K_{k+1} - e^*)$ .
- (iv) First consider a graph  $G$  of  $\mathcal{G}_6$  obtained using  $B_1$  (see Fig. 2). There are two cases.

If  $|U| = 1$ , let  $\{v_1, v_2, v_3\}$  be the vertex-set of  $B_1$  and assume (without a loss of generality) that  $U = \{v_1\}$ . Then  $G$  is a spanning graph of  $H$ , where  $H \in \mathcal{G}_6$  obtained from  $EC(K_{k-1})$  by adding all possible edges between  $\{v_1\}$  and the vertices of the  $K_{k-1}$ . Thereafter, we proceed as in (ii) above by considering  $H_1$  as the path  $P_3$  together with  $2k - 2$  isolated vertices, and  $H_2$  as  $EC(K_k)$  together with an isolated vertex ( $v_2$  or  $v_3$ ). It is easy to see that  $\lambda_1(H_1) = \dots = \lambda_{2k-1}(H_1) = 0$ ,  $\lambda_{2k}(H_1) = 1$  and  $\lambda_{2k+1}(H_1) = 3$ ; and  $\lambda_1(H_2) = \lambda_2(H_2) = 0$  and  $\lambda_3(H_2) = \dots = \lambda_{k+1}(H_2) = \frac{k+2-\sqrt{k^2+4}}{2}$ ,  $\lambda_{k+2} = 2$  and  $\lambda_{k+3}(H_2) = \dots = \lambda_{2k+1}(H_2) = \frac{k+2+\sqrt{k^2+4}}{2}$ . Now, using the Courant–Weyl inequalities, we have

$$\lambda_2(H) \leq \lambda_{2k-1}(H_1) + \lambda_3(H_2) = \frac{k+2-\sqrt{k^2+4}}{2} < \frac{k+3-\sqrt{(k+1)^2+4}}{2}.$$

If  $|U| = 2$ , then  $G$  is a spanning graph of  $PC(K_{k+1})$  and therefore

$$\lambda_2(G) \leq \lambda_2(PC(K_{k+1})) = \frac{k+3-\sqrt{(k+1)^2+4}}{2}.$$

Now, assume that  $G$  is obtained using one of the graphs  $B_2, B_3, B_4$  or  $B_5$ . Let  $\{v_1, \dots, v_5\}$  denote the set of vertices of  $B_i, i = 2, \dots, 5$ . At this level we will consider different cases according to the cardinality of  $U$ . First note that any two vertices of  $B_i, i = 2, \dots, 5$ , have a common neighbor. Therefore  $|U| \leq 2$ .

If  $|U| = 1$ , assume, without a loss of generality, that  $v_1$  is connected to vertices from  $G$ . Let  $v_2$  be a neighbor of  $v_1$  in  $B_i$ . Then  $G$  is the union of  $H_1$  composed of  $B_i - v_1v_2$  together with  $2k-4$  isolated vertices, and  $H_2$  composed of  $EC(K_{k-1})$  together with three isolated vertices  $v_3, v_4$  and  $v_5$ . Using the Courant–Weyl inequalities applied to the Laplacian spectra of  $H_1$  and  $H_2$ , we have

$$a(G) \leq a(H) \leq \lambda_{2k-4}(H_1) + \lambda_5(H_2) = \frac{k+1-\sqrt{(k-1)^2+4}}{2} < \frac{k+3-\sqrt{(k+1)^2+4}}{2}.$$

If  $|U| = 2$ , say  $U = \{v_1, v_2\}$  with  $v_1v_3, v_2v_4 \in E(B_1)$  (we relabel the vertices of  $B_i$  if needed). Then we proceed as above by choosing  $H_1$  composed of  $B_i - \{v_1v_3, v_2v_4\}$  and  $2k-4$  isolated vertices, and  $H_2$  composed of  $EC(K_k)$  together with an isolated vertex  $\{v_5\}$ . Thus

$$a(G) \leq a(H) \leq \lambda_{2k-4}(H_1) + \lambda_5(H_2) = \frac{k+2-\sqrt{k^2+4}}{2} < \frac{k+3-\sqrt{(k+1)^2+4}}{2}.$$

This completes the proof of the lemma.  $\square$

**Theorem 13.** Let  $G$  be a connected graph with odd order  $n = 2k+1 \geq 9$ , algebraic connectivity  $a$ , minimum degree  $\delta$  and domination number  $\beta \geq 3$ . If  $\delta \in \{1, 3, 5\}$  or  $\delta$  is even and  $G \notin \{F_6, F_7, F_8\}$ , then

$$a \leq 2k - 2\beta + \frac{k+3-\sqrt{(k+1)^2+4}}{2}.$$

The bound is the best possible as shown by  $OC(K_{k+1}), PC(K_{k+1})$  and  $PC(K_{k+1}) - e^*$ .

**Proof.** If  $\delta$  is even and  $G \notin \{F_6, F_7, F_8, F_9\}$ , then using Theorem 6,  $2\beta \leq (n-1-\delta)$ . Thus

$$a + 2\beta \leq \delta + 2\beta \leq n-1 < 2k + \frac{k+3-\sqrt{(k+1)^2+4}}{2}.$$

If  $G \equiv F_9$ , the bound is true from Table 2.

If  $\delta = 1$ , the result follows from Theorem 4, Corollary 8 and Lemma 12.

If  $\delta = 3$ , it is known that  $\beta \leq 3n/8$  (see [17] and [12, p. 48]). Thus

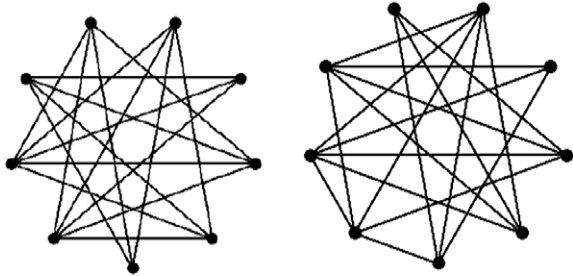
$$a + 2\beta \leq 3 + \frac{3n}{4} < n-1 \quad \text{for all } n \geq 17.$$

For  $n \leq 15$ , we use the maximum possible value for  $\beta$ , denoted by  $\beta^*$ , in a graph on  $n$  vertices with minimum degree  $\delta = 3$ , provided in [8]. Table 3 is obtained. So in fact, we have to check only for  $n = 9$  and  $n = 11$ .

Using McKay's program *nauty* (available at "http://cs.anu.edu.au/~bdm/nauty"), we generated all graphs on  $n = 9$  vertices with  $\delta = 3$  and maximum degree at most  $\Delta = 6$  (if  $\Delta = 7$ , necessarily  $\beta = 2$ , and therefore the inequality is strict). There are exactly 41 113 such graphs, among which there are exactly 484 with  $\beta = 3$ . Over all these 484 graphs, the algebraic connectivity is at most  $a = 2.4604154$  which is reached for only two graphs (presented in Fig 5), while the corresponding value of the bound is approximately 2.8074176.

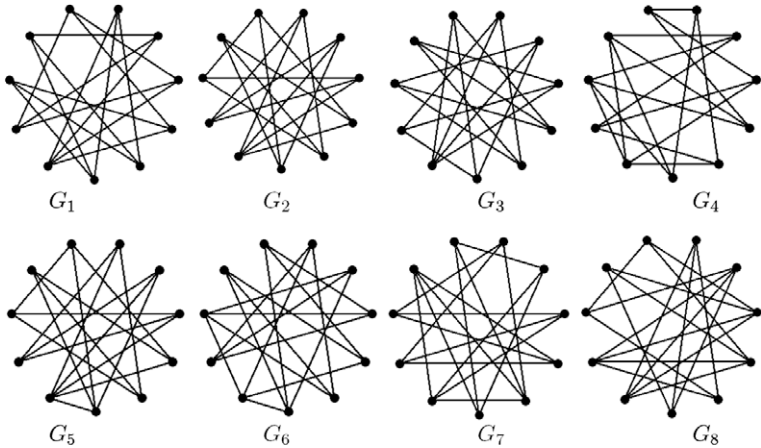
**Table 3**  
Values of  $\delta + 2\beta^*$  for  $n = 9, 11, \dots, 15$  and  $\delta = 3$ .

$n$	9	11	13	15
$\delta + 2\beta^*$	9	11	11	13



**Fig. 5.** The two graphs that maximize  $a$  for  $n = 9, \delta = 3$  and  $\beta = 3$ .

For  $n = 11$ , we have to check for graphs with  $\delta = 3$  and  $\beta = 4$ . Using *nauty*, we generated all graphs with minimum degree  $\delta = 3$ , maximum degree  $\Delta \leq 7$  (since  $\Delta \geq 8$  implies  $\beta \leq 3$ ) and size  $17 \leq m \leq 31$  (due to the inequality  $\beta \leq n + 1 - \sqrt{2m + 1}$  [12, p. 55]). There are exactly 205 662 831 such graphs, among which only 8 have domination number  $\beta = 4$  and they are given in Fig. 6. We did not explore all the 205 662 831 graphs. We first computed the domination for the graphs on up to 21 edges. Since any graph  $G$  on  $m = 22$  edges with minimum degree 3 contains at least one edge  $uv$  such that  $d(u) \geq 4$  and  $d(v) \geq 4$ , so can be obtained from a graph  $G'$  on  $m - 1 = 21$  edges with minimum degree 3, by adding an edge. Since adding an edges does not increase the domination number,  $\beta(G) \leq \beta(G')$ . Among all the 1 225 809 graphs on 11 vertices and 21 edges with  $\delta = 3$  and  $\Delta \leq 7$ , there was no graph with  $\beta = 4$ . Thus, it is so for  $m = 22$  and recursively for  $m = 23, \dots, 31$ .



**Fig. 6.** All the graphs on  $n = 11$  vertices with  $\delta = 3$  and  $\beta = 4$ .

**Table 4**  
Values of  $a(G_i)$  for  $i = 1, 2, \dots, 8$ .

$i$	1	2	3	4	5	6	7	8
$a(G_i)$	0.7382	1.1864	1.3446	1.5013	1.3075	1.3937	0.7382	1.6672

The algebraic connectivities of the eight graphs (given in Fig. 6) on  $n = 11$  vertices with minimum degree  $\delta = 3$  and domination  $\beta = 4$  are given in Table 4. Thus, the bound is not reached for any of these graphs.

**Table 5**  
Values of  $\delta + 2\beta^*$  for  $n = 11, 13, \dots, 19$  and  $\delta = 5$ .

$n$	11	13	15	17	19
$\delta + 2\beta^*$	11	11	13	15	17

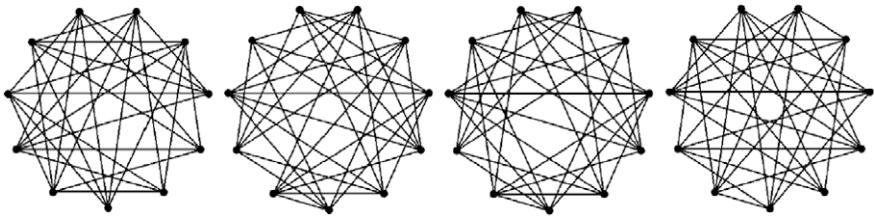


Fig. 7. The four graphs with  $n = 11, \delta = 5$  and  $\beta = 3$  that maximize  $a$ .

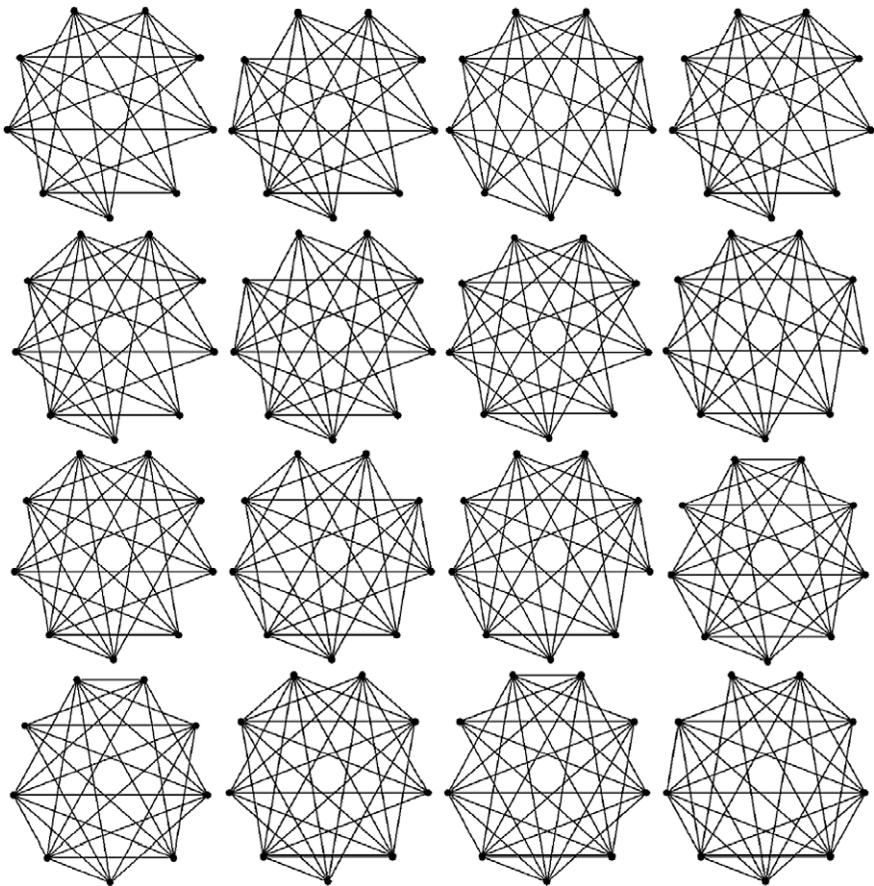


Fig. 8. The 16 graphs on  $n = 9$  vertices with  $a = \delta = 5$  and  $\beta = 2$ .

If  $\delta = 5$ , it is proved in [19] that  $\beta \leq 5n/14$ . Thus

$$a + 2\beta \leq 5 + \frac{5n}{7} \leq n - 1 \quad \text{for all } n \geq 21.$$

Therefore, the bound is true, with strict inequality, for all  $n \geq 21$ . For  $n \in \{7, 9, \dots, 19\}$ , we use the values of  $\beta^*$  from [8] for  $n = 9, \dots, 15$ , and an upper bound on  $\beta^*$  for  $n \in \{17, 19\}$  computed using the following formula from [7],  $\beta^* \leq \min\{p, g_p = 0\}$ , where  $g_p$  is defined by

$$g_0 = n \quad \text{and} \quad g_{p+1} = \left\lfloor g_p \left(1 - \frac{\delta + 1}{n - p}\right) \right\rfloor. \quad (13)$$

The corresponding values are given in Table 5. Since for  $n = 9$ ,  $\beta^* = 2$  and the desired inequality is strict for  $n \in \{13, 15, 17, 19\}$ , we have to check only for graphs on  $n = 11$  vertices with  $\delta = 5$  and  $\beta = 3$ . Note in addition to these conditions, if  $\Delta \geq 8$  then  $\beta \leq 2$ . So using *nauty*, we enumerated all graphs on  $n = 11$  vertices with  $\delta = 5$  and  $\Delta \leq 7$ . There are exactly 3 982 767 graphs satisfying these conditions, 2098 of which have  $\beta = 3$ . Among these 2098 graphs, the algebraic connectivity is maximum for the four presented in Fig. 7 for which  $a = 0.26795$ , while the value of the bound corresponding to  $n = 11$  is 0.83772.

This completes the proof.  $\square$

Note that the condition  $\beta \geq 3$  in Theorem 13 is necessary. Indeed, when exploring graphs on  $n = 9$  vertices with minimum degree  $\delta = 5$ , we found exactly 16 graphs (Fig. 8) with  $\beta = 2$  and  $a = \delta = 5$ . For these graphs the bound from Theorem 13 does not hold.

We are convinced that Theorem 13 is true for all values of the minimum degree, however we do not yet have the proof, so we close with the following conjecture.

**Conjecture 14.** Let  $G$  be a connected graph with odd order  $n = 2k + 1 \geq 9$ , algebraic connectivity  $a$ , minimum degree  $\delta$  and domination number  $\beta \geq 3$ . If  $G \notin \{A_3, A_4, F_6, F_7, F_8\}$ , then

$$a \leq 2k - 2\beta + \frac{k + 3 - \sqrt{(k + 1)^2 + 4}}{2}$$

with equality if and only if  $G$  is  $OC(K_{k+1})$ ,  $PC(K_{k+1})$  or  $PC(K_{k+1}) - e^*$ .

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